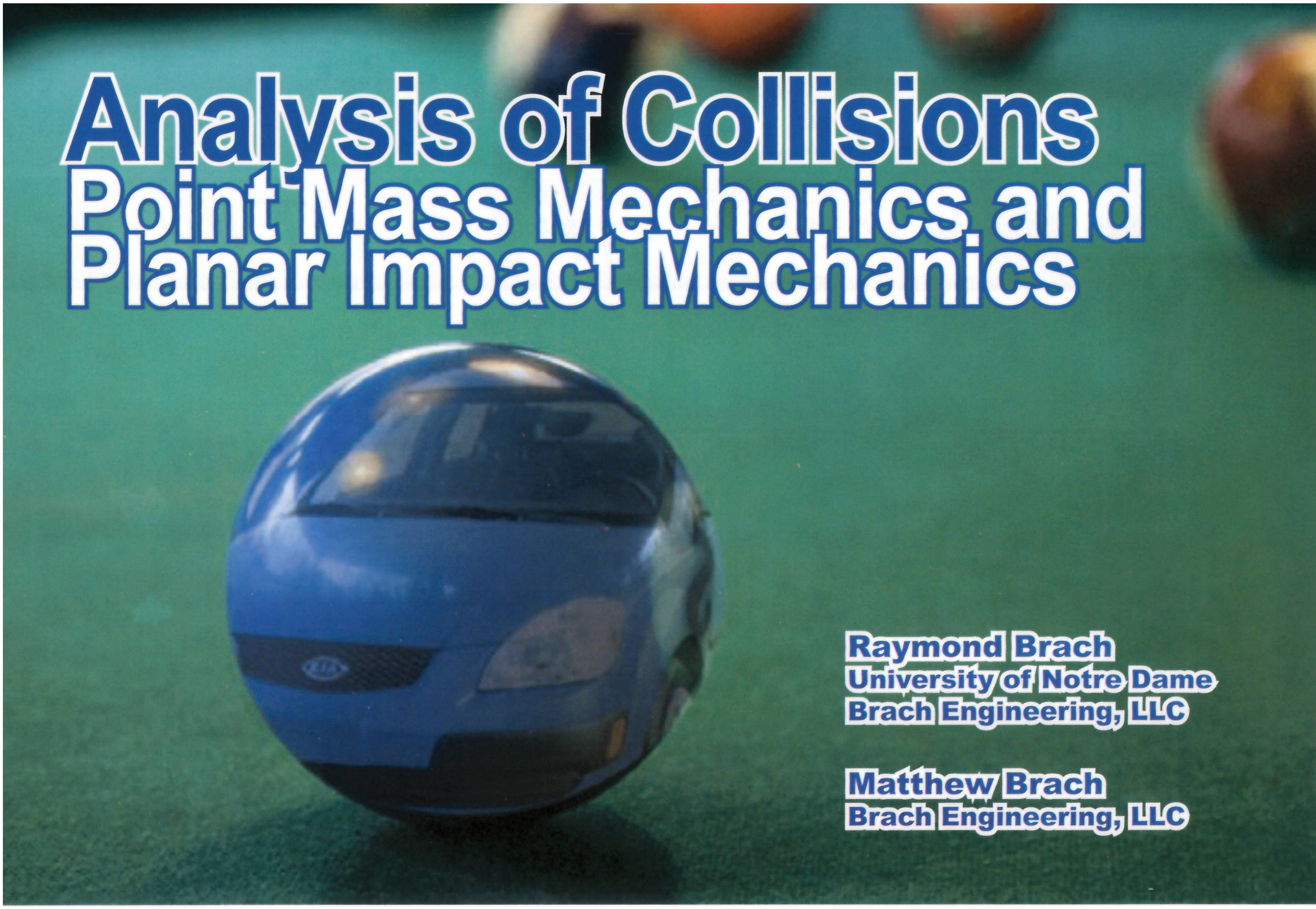

Analysis of Collisions Point Mass Mechanics and Planar Impact Mechanics

Raymond Brach
University of Notre Dame
Brach Engineering, LLC

Matthew Brach
Brach Engineering, LLC

Introduction

A method used frequently in the reconstruction of collisions has been given the name *conservation of momentum*. More specifically sometimes it is referred to as the *conservation of linear momentum* (COLM). This article discusses the assumptions behind COLM, how it is applied with the skid-to-stop formula to reconstruct crashes and some problems that can arise if conservation of linear momentum is not applied carefully. A more general form of COLM is presented including concepts of restitution, force, impulse and PDOF. Energy is discussed, particularly if energy is or is not conserved in a collision.

Part 2 of this article covers planar impact mechanics. This is a more general method of analyzing and reconstructing collisions that takes rotational velocities and rotational momentum of the vehicles into account. Comparisons of reconstruction calculations using planar impact mechanics are made to reconstruction calculations using point mass mechanics. Planar impact mechanics gives more accurate results since it is more rigorous. The improvement in accuracy is illustrated in the comparisons using examples where significant differences occur.

Analysis of Collisions, Part 1: Point Mass Mechanics

Conservation of Linear Momentum: Suppose an intersection collision between two vehicles occurs (Figure 1) where the following are known or can reasonably be assumed to be true based on an investigation:

1. the initial directions of motion of each vehicle at the beginning of contact, that is, angles θ_1 and θ_2,
2. the final directions of motion of each vehicle at separation from impact, that is, angles Θ_1 and Θ_2,
3. the travel distances, d_1 and d_2, from separation to rest of each vehicle, where

$$d_1 = \sqrt{x_1^2 + y_1^2} \tag{1}$$

and

$$d_2 = \sqrt{x_2^2 + y_2^2} \tag{2}$$

4. the frictional drag coefficients, f_1 and f_2, or their equivalent values, of the vehicles from impact to rest.

Note that Figure 1 shows two coordinate systems, *x-y* and *n-t*. The *x-y* system is one fixed to the ground or roadway. The *n-t* system is orientated along a common crush surface of the vehicles in their crash positions, referred to as the intervehicular crush surface. The angle between these coordinate systems is Γ. The utility of this second coordinate system will become evident later.

Under such circumstances (and other conditions to be discussed later), the skid-to-stop formula can be used to provide the final collision speeds at separation, V_1 and V_2 (final impact

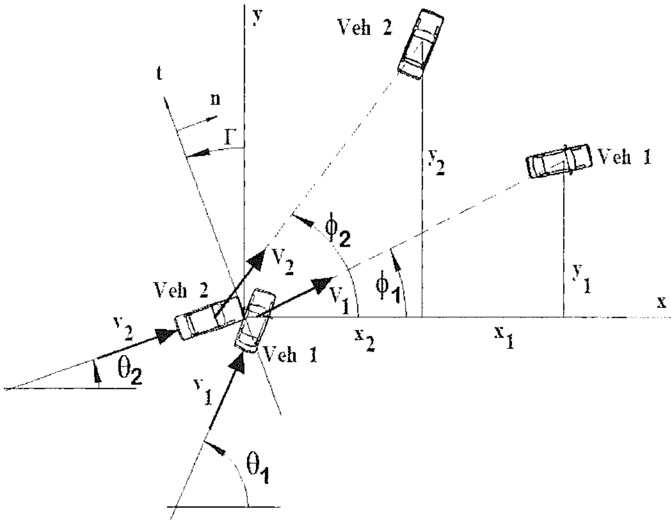

Figure 1

velocity symbols are capitalized). That is:

$$V_1 = \sqrt{2gf_1d_1} \tag{3}$$

And

$$V_2 = \sqrt{2gf_2d_2} \tag{4}$$

So at this point the final impact velocities, V'_1 and V'_2, are known. Knowing the final velocities, conservation of linear momentum can be used to compute the initial impact velocities, v_1 and v_2 (initial impact velocity symbols are lower case), of both vehicles. From conservation of momentum in the x direction:

$$m_1v_1\cos\theta_1 + m_2v_2\cos\theta_2 = m_1V_1\cos\phi_1 + m_2V_2\cos\phi_2 \tag{5}$$

and from conservation of momentum in the y direction

$$m_1v_1\sin\theta_1 + m_2v_2\sin\theta_2 = m_1V_1\sin\phi_1 + m_2V_2\sin\phi_2 \tag{6}$$

Equations 5 and 6 are two linear algebraic equations in the unknown initial velocities, v_1 and v_2. They can be solved using methods of algebra; the solution is given in Appendix A (Eq A1 and A2).

Since most readers already are familiar with COLM, a typical application is presented using the skid-to-stop formulas (Eq 3 and 4) combined with the solution of the conservation of linear momentum (Eq 5 and 6).

Example 1: Suppose two vehicles collide in an intersection as shown in Fig 2 with Veh 1 originally eastbound and Veh 2 northbound. Both vehicles weigh 2500 lb. After impact the center of gravity of Veh 1 travels x_1 = 42.0 ft and y_1 = 52.0 ft to rest. The center of gravity of Veh 2 travels x_2 = 32.0 ft and y_2 = 36.0 ft to rest. From impact to rest, both vehicles remain on the same flat road surface with a frictional drag coefficient estimated to be f = 0.70.

The initial speeds of the vehicles can be found using Eq 3, 4, 5 and 6. First, Eq 1 and 2 give d_1 = 66.84 ft and d_2 = 48.17 ft. Next, Eq 3 and 4 give V'_1 = 54.9 ft/s (37.4 mph) and V'_2 =

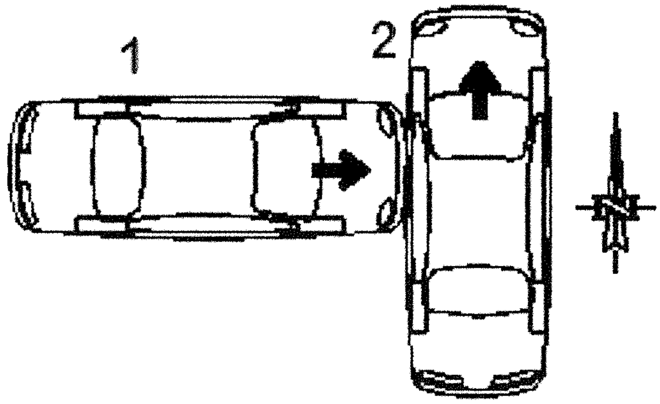

Figure 2

46.6 ft/s (31.8 mph). Then the solutions (from Appendix A) of Eq 5 and 6 give v_1 = 65.4 ft/s (44.6 mph) and v_2 = 77.5 ft/s (52.8 mph).

The equations from COLM and skid to stop can provide useful results but are only part of the information mechanics can provide. When vehicles collide at high speeds (closing speeds of roughly 20 mph or more), they typically are momentarily "stuck together" at their contact surface at the instant of separation; this phenomenon is ordinarily referred to as the common velocity condition [1]. To determine if the solution of the above equations meets the common velocity condition, additional equations of mechanics must be developed. In particular, the concepts of restitution and relative sliding over the contact surface at separation are examined. When studying collisions it is common to define what is called a coefficient of restitution, e, as

$$e = -\frac{V_{2n} - V_{1n}}{v_{2n} - v_{1n}} \tag{7}$$

where V'_{1n} is the final velocity of Veh 1 in the direction of the normal coordinate, n, in Fig 1, V'_{2n} is the final velocity of Veh 2 in the direction of the normal coordinate, n, v_{1n} is the initial velocity of Veh 1 in the direction of the normal coordinate, n, and v_{2n} is the initial velocity of Veh 2 in the direction of the normal coordinate, n, in Fig 1. The numerator in Eq 7 represents the parting velocity (rebound velocity) and the denominator represents the closing velocity (or approach velocity). The negative sign comes into play because in an impact, the direction of the relative normal (perpendicular) velocity reverses as a result of the impact. All of the velocity components in Eq 7 are measured normal to a common intervehicular crush surface defined by the angle Γ as shown in Fig 1. For all point mass collisions (not just vehicle collisions), the coefficient of restitution is bounded, that is, $0 \le e \le 1$. The common velocity condition (at the instant of separation) normal to the intervehicular crush surface means that $e = 0$. In words, this means that $V'_{1n} = V'_{2n}$ and the vehicles do not bounce apart, or in more formal terms, the collision is perfectly inelastic. For most high speed vehicle collisions, e typically is less than 0.3 [2] and, experimental data show that it is typically very close to zero.

Once the preimpact velocities of the vehicles are known, many more results from the example collision can be found from the mechanics equations of a point mass collision [2]. These results include the magnitude and directions of impulses, the collision kinetic energy loss, each vehicle's ΔV, the PDOF for each vehicle, etc. The equations for such quantities are given in Appendix A. More is covered in what follows.

The assumptions that underlie the COLM and impact equations are very important and must always be checked to make sure they are satisfied for each application. The basic assumptions for point mass impact theory are [3]:

1. The impact represents a single dynamic contact, taking place over a short duration.
2. Forces other than the vehicle-to-vehicle contact force and impulses of forces other than the contact impulse are negligible.
3. Rotational motion (angular velocities) of the masses is negligible.
4. The initial velocities are known and final velocities are unknown (in some special cases the opposite can be true).
5. The deformation is localized and small compared to the size of the bodies (that is, the center of gravity location is not significantly changed by damage).
6. During the contact duration, position changes are negligibly small, velocity changes are instantaneous and accelerations are large.
7. The effects of the normal (crush) and tangential (sliding, shearing, entanglement, etc.) contact processes are known and are described by two impact coefficients.
8. The masses (weights) remain essentially constant.

In accident reconstruction, one of the most important assumptions is number 2. Forces such as the tire-ground forces must be small compared to the force acting over the intervehicular contact surface. This is why the impact equations should be applied to high speed collisions only. With lower contact forces between each vehicle, tire-road frictional forces can be significant for low speed collisions and their collisions must be treated in a different fashion [2].

The use of the skid-to-stop formula also involves assumptions. One is that each of the vehicles travels from impact to rest with all wheels locked or, at least, with uniform deceleration. Another is that the change in heading of each vehicle (due to impact rotational velocity changes) has a negligible effect on the deceleration.

Example 1 (continued): The coefficient of restitution for the collision of Example 1 was not needed in the solution of Eqs 5 and 6, but once the initial velocities are known it can be found. Its value can be calculated using Eq 7 and other equations from Appendix A. For a crush surface angle of $\Gamma = 0°$ (see Figs 1 and 2) the coefficient of restitution is:

$$e = -\frac{V_{2n} - V_{1n}}{v_{2n} - v_{1n}} = -\frac{24.57 - 18.64}{43.21 - 0.0} = -0.14$$

Since e must be greater than zero for realistic collisions, a negative value of e should alert the reconstructionist that something is amiss. A negative coefficient of restitution signifies that the vehicles passed through each other [4] which is physically impossible. At this point it is a good idea to check the assumptions. Perhaps, for example, the motion from impact to rest of one or both vehicles does not satisfy the locked-wheel assumption. This is implied by e being negative and Veh 1 seemingly passing through Veh 2 (see Fig 1). Suppose the wheels of Veh 1 were not locked for the entire postimpact distance, or that two of its wheels were locked due to damage and the other two were free to roll, so that $f_1 = 0.56$ instead of 0.70. For this alternative value of f_1, the skid to stop equations give $V'_1 = 49.28$ ft/s (33.6 mph). The solution of Eq 5 and 6 now gives $v_1 = 61.9$ ft/s (42.2 mph) and $v_2 = 73.1$ ft/s (49.9 mph). More importantly, the coefficient of restitution, e, is equal to 0 and the results of the reconstruction become physically realistic.

In fact, the approach to the reconstruction that provided these new results was to find the value of f_1 that gave a value of $e = 0$. This was done using the search/optimization feature of a spreadsheet. The full reconstruction results are illustrated in Fig 3.

Point Mass Impact Mechanics:

It was seen above that impact mechanics, even for point masses, is broader, or more general, than COLM. To gain a full understanding, some of the basic concepts of impact mechanics must be understood. Typically, the concept of a force is a beginning point. When two objects come into contact, each transmits an equal and opposite dynamic force, F, to the other (this follows from Newton's Third Law). For an impact the contact force starts from zero (at the beginning of contact), reaches some peak value and then drops to zero when the bodies separate. The profile of such an intervehicular force is shown in Fig 3. The area under the force-time curve, shown shaded in Fig 4, is defined as the impulse and here is given the symbol, P. An impulse is an accumulation of the action of a force over time. If needed, an average value of the force can be found if the duration and impulse are known. For example, Fig 4 shows the average of that intervehicular force, F_{avg}.

Newton's Second Law can be stated in terms of impulses and

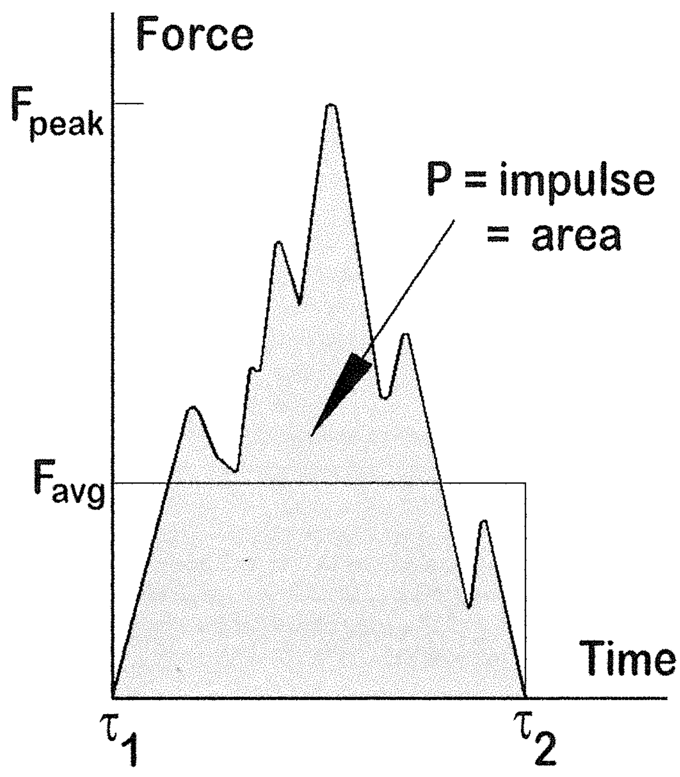

Figure 4

Example1COL.xls
4/13/07
ver. 1.2

Skid To Rest Collision Reconstruction

>			UNIT CONVERSION
ID:			
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frictional drag factors	
f_1:	0.56
f_2:	0.70

vehicle weights, lb	
W_1	2500.0
W_2	2500.0
mass, lb-s^2/ft	
m_1	77.70
m_2	77.70
g	32.17
conv	1.47
det	6.04E+03
R1	4.81E+03
R2	5.68E+03
ΔV_1:	49.25
ΔV_2:	49.25
coeff, e	0.00
coeff, μ	-1.24
coeff, μ_c	-1.18

postimpact skid distances, ft					
d_1:	66.84	x_1	42.0	y_1	52.0
d_2:	48.17	x_2	32.0	y_2	36.0

postmpact speeds, ft/s & mph			
V_1:	49.25	33.58	
V_2:	46.58	31.76	
V_{1x}	30.95	V_{1y}	38.32
V_{2x}	30.95	V_{2y}	34.81

postimpact angles	deg	rad
ϕ_1	51.1	0.89
ϕ_2	48.4	0.84

preimpact speeds, ft/s & mph			
v_1:	61.89	42.2	
v_2:	73.13	49.9	
v_{1x}	61.89	v_{1y}	0.00
v_{2x}	0.00	v_{2y}	73.13

preimpact angles	deg	rad
θ_1	0.0	0.00
θ_2	90.0	1.57

System Energy	
Initial, ft-lb	356,608.0
Final, ft-lb	178,542.4
Loss	178,065.6
%	49.9

Γ	0.0		
$\sin\Gamma$	0.000	$\cos\Gamma$	1.000
v_{1n}	61.89	v_{1t}	0.00
v_{2n}	0.00	v_{2t}	73.13
V_{1n}	30.95	V_{1t}	38.32
V_{2n}	30.95	V_{2t}	34.81

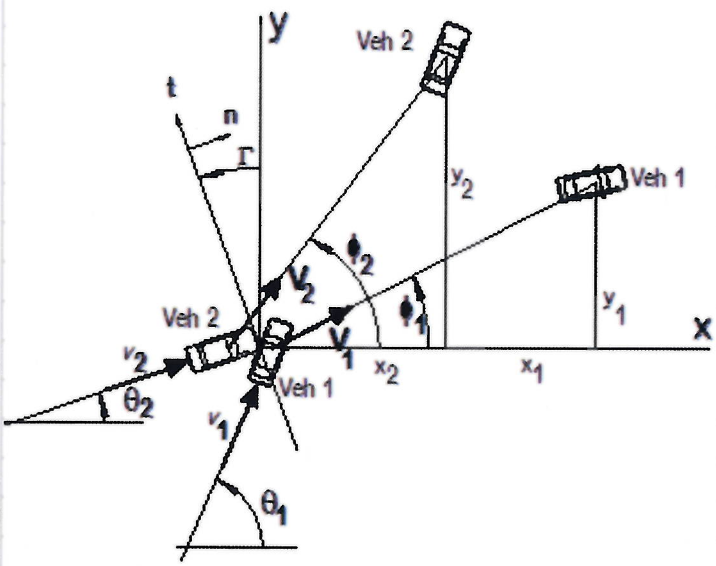

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Figure 3

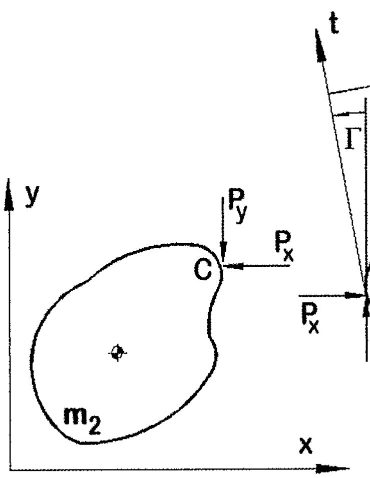


Figure 5

changes in momentum [5]. For example, from Fig 5, the impulse component, P_x, in the x direction is equal to the change in momentum of mass, m_1, in the x direction. In mathematical form, this is:

$$m_1 V_{1x} - m_1 v_{1x} = P_x \quad (8)$$

Recall that capital V is a symbol that represents a final impact velocity and lower-case v is a symbol that represents an initial impact velocity. Because of the equal and opposite nature of forces (and impulses), a similar equation can be written for m_2, namely:

$$m_2 V_{2x} - m_2 v_{2x} = -P_x \quad (9)$$

The same holds true for the y components, giving two more equations.

$$m_1 V_{1y} - m_1 v_{1y} = P_y \quad (10)$$

and

$$m_2 V_{2y} - m_2 v_{2y} = -P_y \quad (11)$$

In Fig 5 the x-y coordinate system can be thought of as one attached to the ground (road), whereas, the n-t coordinate system can be thought of as one attached to a common crushed contact surface of the masses (vehicles). Using trigonometry and the angle Γ, Appendix A provides equations that permit the impulses to be expressed as components in the n-t coordinate system as well as components in the x-y system.

In a broad sense, energy in a collision is conserved. Some of the energy in a collision is converted to sound, some is converted to damage (crushing, displacing, scraping, gouging, sliding etc.) of the metal and plastic materials. Some energy even can be diverted into injuries of the occupants. But, to a reconstructionist, the most important aspect of collision energy is that in all vehicle collisions, some percentage of the combined initial kinetic energy of the two vehicles (the *vehicle system* kinetic energy) is lost. This kinetic energy loss can be calculated by computing the initial kinetic energy of both vehicles, computing the final kinetic energy of both vehicles and subtracting them. Another way is to use the equation from Appendix A:

$$KE_{loss} = \frac{1}{2}\bar{m}(v_{2n} - v_{1n})^2 (1+e)\left[(1-e) + 2\mu r - (1+e)\mu^2\right] \quad (12)$$

where

$$\bar{m} = \frac{m_1 m_2}{m_1 + m_2} \quad (13)$$

$$r = \frac{v_{2t} - v_{1t}}{v_{2n} - v_{1n}} \quad (14)$$

e is the coefficient of restitution and μ is referred to as the impulse ratio. It is equal to the ratio of the tangential impulse component and the normal impulse component:

$$\mu = P_t / P_n \quad (15)$$

The quantity μ is sort of an equivalent impulse friction coefficient, but has some important differences from a friction (force) coefficient [2]. The total impulse, P, can be found from its components:

$$P = \sqrt{P_x^2 + P_y^2} = \sqrt{P_n^2 + P_t^2} \quad (16)$$

Once P is found, the direction, a, of its line of action can be established using trigonometry from:

$$a = \tan^{-1}(P_y / P_x) \quad (17)$$

Knowing a, the principal direction of force, PDOF, can be found [6].

Example 1 (continued): Some of the other features of the reconstructed collision can be examined. For example, using Eq 8 and 10 and values from Fig 3, the impulse components have the values:

$$P_x = \frac{2500}{g}(30.95 - 61.90) = -2405.19,\ lb - s$$

$$P_y = \frac{2500}{g}(38.32 - 0.0) = 2977.93,\ lb - s$$

which produces an impulse magnitude of P = 3827.93 lb-s. In this example, the intervehicular contact surface is parallel to the y axis, the x-y and n-t axes are coincident and Γ = 0. Consequently, $P_x = P_n$ and $P_y = P_t$. In addition, using Eq 12, the kinetic energy loss is 178,102.7 ft-lb, which is 49.9% of the total initial kinetic energy of the vehicles. Since P_x and P_y are known, the PDOF of each vehicle can be determined. The angle of the impulse is

$$a = \tan^{-1}(P_y / P_x) = \tan^{-1}(-1.238) = 128.9°$$

This means that $pdof_1$ = 51.1° and $pdof_2$ = -38.9°.

Additional equations can be written, more examples can be worked, but at some point, a very important question must be asked. Is point mass mechanics the best way to model or analyze a collision? In other words, is there a better, more accurate method? The answer to the second question is yes. A method called *planar impact mechanics* is available [7, 8, 9] that can model collisions, including the effects of rotational momentum. Surprisingly, the application of the method isn't any more difficult than point mass mechanics. An explanation of planar impact mechanics follows in Part 2.

Analysis of Collisions, Part 2: Planar Impact Mechanics The words *planar impact mechanics* refer to the analysis of the mechanics (motion and forces) of an impact between two rigid bodies that takes place in a plane. Figure 6 shows two rigid bodies (vehicles) in contact and with a common point C. It shows them separated to display the physical action of each on the other. The term *rigid body* does not imply that the bodies are not deformable, but rather that they are not point masses; their mass extends beyond their centers of gravity. In other words they possess a moment of inertia. In this case, each moment of inertia is about a vertical axis through the center of gravity and corresponds to rotational motion referred to as yaw rotation. By methodically applying Newton's Second Law in the form of impulse and momentum, a set of impact equations can be developed that can be used to analyze a vehicle collision taking into account the change in yaw velocity due to the impact. What is remarkable about these equations is that they can be solved using algebra, that is without resorting to a computer. But more on this later.

In this Part of the article, only the collision is analyzed. Since postimpact motion of vehicles in actual practice seldom takes placed under uniformly locked wheel conditions, the motion following the collision typically is analyzed separately from the collision using a computerized vehicle dynamic simulation program [9, 10, 11, 12].

The assumptions that underlie planar impact mechanics are identical to those for the point mass impact mechanics with one exception. For planar impact mechanics, Assumption 3 (given above for point mass mechanics) concerning rotational motion (angular velocities) is dropped.

As in Part 1, the impact problem is treated as one where the initial velocities are known and the final values are to be computed. The initial velocities are v_{1x}, v_{1y}, ω_1, v_{2x}, v_{2y} and ω_2. The final velocities are V_{1x}, V_{1y}, Ω_1, V_{2x}, V_{2y} and Ω_2. Accident reconstructionists would usually like to be able to solve the impact problem the opposite way (determine the initial velocities that cause certain final velocities). Unfortunately, the mathematics will not allow that [4]. To set up the proper equations including the effects of rotation, it is necessary to define a common point C located on the intervehicular crush surface (see Fig 6) called the *impact center*. Point C is the point of action of the impulse, *P*. It lies on the contact surface and its location is estimated from viewing the damage to the vehicles. Point C is the point through which the PDOF acts. It is specified (located) by the distances d_1, and d_2 and the angles φ_1 and φ_2 relative to the centers of gravity of the vehicles (see Fig 6). The heading angle of each vehicle during impact relative to the *x-y* coordinate system is given by θ_1 and θ_2, respectively, for vehicle 1 and vehicle 2. These angles are not exactly the same as the initial directions of motion as in the point impact case covered in Part 1. This is because, each vehicle can have an initial angular velocity, ω, and could even be sliding sideways (although typically this is not the case).

As in Part 1, Newton's Second law in the form of impulse and momentum can be applied [5]. A change in rotational momentum for each vehicle must now be included. This is done here using *n-t-θ* coordinates (rather than *x-y-θ*) for convenience. Applying Newton's law for each coordinate and for each vehicle gives 6 total impulse and momentum equations. These are:

$$m_1(V_{1n} - v_{1n}) = P_n \quad (18)$$

$$m_1(V_{1t} - v_{1t}) = P_t \quad (19)$$

$$I_1(\Omega_1 - w_1) = d_c P_n - d_d P_t \quad (20)$$

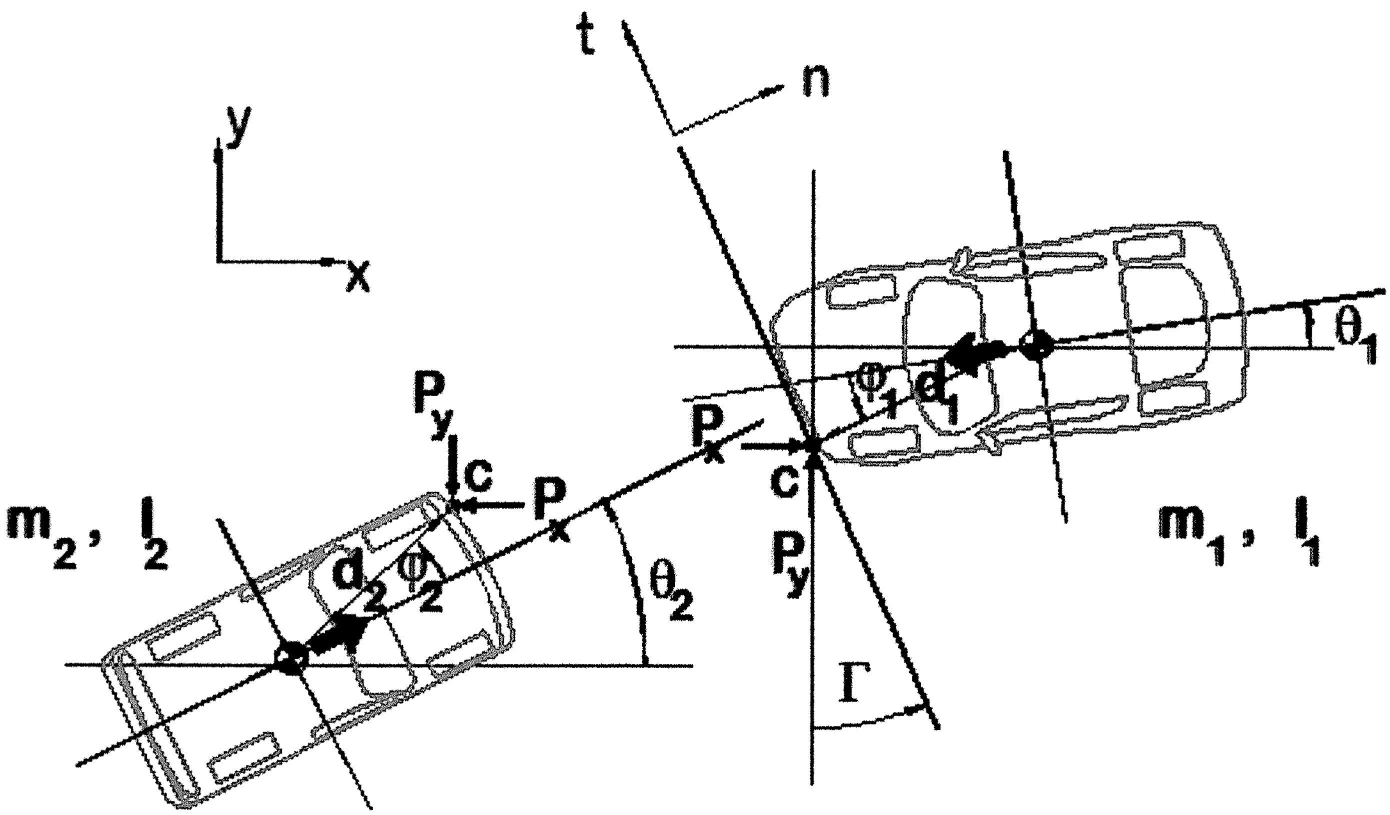


Figure 6

$$m_2(V_{2n} - v_{2n}) = -P_n \quad (21)$$

$$m_2(V_{2t} - v_{2t}) = -P_t \quad (22)$$

$$I_2(\Omega_2 - \omega_2) = d_a P_n - d_b P_t \quad (23)$$

As before, capital symbols represent final values and lower case symbols represent initial values; this notation includes the rotational velocity symbols, Ω and ω. The velocity components v_{1n}, v_{1t}, v_{2n}, v_{2t}, V_{1n}, V_{1t}, V_{2n} and V_{2t} are those of the center of mass of the vehicles. The distances d_a, d_b, d_c and d_d are defined in Appendix B. There are 6 unknown final velocity components, V_{1n}, V_{1t}, Ω_1, V_{2n}, V_{2t} and Ω_2. The impulse components, P_n and P_t, also are unknown. This means there are 8 unknowns and 6 equations. Two more equations are needed. Two impact coefficients can be defined as was done in the point mass problem. The coefficient of restitution is:

$$e = -\frac{V_{Crn}}{v_{Crn}} \quad (24)$$

where V_{Crn} is the normal component of the rebound velocity at point C and v_{Crn} is the normal component of the closing velocity at point C (these are expressed more fully in Appendix B). The second impact coefficient is the ratio of impulses,

$$\mu = P_t / P_n \quad (25)$$

Equations 18 through 25 form a set of linear algebraic equations in the 8 unknowns. A remarkable feature of these equations is that their solutions can be stated without the use of a computer [2]. The solution equations are:

$$V_{1n} = v_{1n} + \bar{m}(1+e)v_{rn}q / m_1 \quad (26)$$

$$V_{1t} = v_{1t} + \mu\bar{m}(1+e)v_{rn}q / m_1 \quad 27)$$

$$V_{2n} = v_{2n} - \bar{m}(1+e)v_{rn}q / m_2 \quad (28)$$

$$V_{2t} = v_{2t} - \mu\bar{m}(1+e)v_{rn}q / m_2 \quad (29)$$

$$\Omega_1 = \omega_1 + \bar{m}(1+e)v_{rn}(d_c - \mu d_d)q / I_1 \quad (30)$$

$$\Omega_2 = \omega_2 + \bar{m}(1+e)v_{rn}(d_a - \mu d_b)q / I_2 \quad (31)$$

The quantity v_{rn} is the closing velocity which now depends also on the initial angular velocity:

$$v_{rn} = (v_{2n} - d_a\omega_2) - (v_{1n} - d_c\omega_1) \quad (32)$$

Other variables, such as q, are given in Appendix B. All of this means that given the initial velocities of any two vehicles' physical properties (their masses, yaw inertias and dimensions), their collision configurations, their damage profiles (to determine Point C and the angle Γ), their impact coefficients (e and μ) and their initial velocities that their final velocities can be calculated. Once a solution is obtained, all of the mechanics of the collision

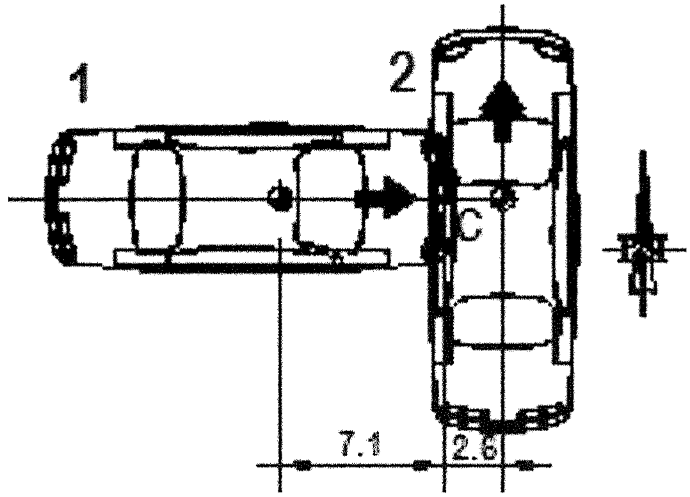


Figure 7

is known. This includes the collision kinetic energy loss, the velocity change, ΔV, of each vehicle, the PDOF of each vehicle and so on.

Example 1 (revisited): Consider again the above example where an eastbound vehicle collides into the driver's side of a northbound vehicle (Fig 7). A new analysis is made here using planar impact mechanics that includes rotational effects. To facilitate a later comparison, the initial velocities found from the point mass analysis are used as the initial conditions along with common velocity conditions. The final velocities, impact kinetic energy loss and vehicle ΔV values from the point mass solution and the planar impact mechanics solutions will be compared. Point C in Fig 7 is an estimate of the impact center. As determined from the vehicle dimensions and crush, $d_1 = 7.1$ ft and $d_2 = 2.6$ ft. The crush surface and impact configuration give $\Gamma = 0°$. Based on the diagram in Fig 5 and Fig 7, $\varphi_1 = 0°$ and $\varphi_2 = 90°$. The orientations of the vehicles at impact determine that $\theta_1 = 180°$ and $\theta_2 = 90°$. The yaw moments of inertia are $I_1 = I_2 = 1575$ ft-lb-s^2. The initial velocities are: $v_{1n} = 61.9$ ft/s, $v_{2n} = 0.0$ ft/s, $v_{1t} = 0.0$ ft/s and $v_{2t} = 73.1$ ft/s. Based on information from the accident investigation, it is assumed any initial yaw velocity is small, so that $\omega_1 = \omega_2 = 0$.

Figure 8 contains a spreadsheet version of the solution equations of planar impact mechanics (Eq 27 - 32) for these input values. Table 1 compares point mass results with the planar impact mechanics results in Fig 8. Relatively large differences occur in many of the final quantities. In particular, the final angular velocities from the planar impact mechanics solution are significant, but ignored in the point mass solution. A large difference exists in the kinetic energy loss. This difference is due, in part, to neglecting the rotational kinetic energy in the point mass solution, and also due to the decrease in the tangential impulse when the freedom to rotate exists. The ΔV values are lower from the planar impact mechanics solution and the PDOFs are changed significantly. The accuracy of these quantities is important when assessing injury potential and when using the ΔV values of an EDR in a reconstruction.

Example 2: Consider an example identical to the Example 1 except where the impact center is now over the driver's side rear wheel of Veh 2 as shown in Fig 9. All other conditions are the same as the last part of Example 1. Here, d_2 and φ_2 are different from above and are 6.9 ft and 159.4°, respectively.

Results of the planar impact mechanics analysis for this example are shown in Fig 10. Some important results are included in Table 1 for comparison to previous results. Comparison between the planar impact mechanics analysis results from Example 1 with Example 2 shows significant differences. These differences are most evident in the collision kinetic energy loss and the ΔV values but also show up in some of the final velocities and PDOF vales as well. Conceivably, the collision in Fig 9 could be analyzed or reconstructed using point mass mechanics. In that case, it is legitimate to compare the results of the point mass solution of Example 1 to the planar impact mechanics results of Example 2. Table 1 shows again that the differences are dramatic. Among other differences, the point mass ΔV values change from 49.3 mph to 18.4 mph and the point mass energy loss changes from 50% to 19%.

Table 1. Comparison of results for Example 1 from the Point Mass Solution and the Planar Impact Mechanics Solution

v_1 = 61.90 ft/s, v_2 = 73.13 ft/s

	Point Mass Solution	**Planar Impact Mechanics Solutions**	
	Example 1	**Example 1**	**Example 2**
V_{1n}', ft/s	30.95	30.95	49.17
V_{1t}', ft/s	38.32	15.17	13.25
V_{2n}', ft/s	30.95	30.95	12.73
V_{2t}', ft/s	34.81	57.97	59.89
Ω_1, deg/s	-	304.5	265.9
Ω_2, deg/s	-	111.5	323.3
KE_{loss}, ft-lb	178103 (50%)	117555 (33%)	68257 (19.1%)
ΔV_1, ft/s	49.3	34.5	18.4
ΔV_2, ft/s	49.3	34.5	18.4
P_n, lb-s	-2404	-2401	-969
P_t, lb-s	3318	1179	1029
$pdof_1$, deg 51.1	26.1	46.1	
$pdof_2$, deg -38.9	-63.9	-43.9	

Table 1

Conclusions

It is clear from the above examples that not only are rotations important, but that the effects of an offset in the vehicle alignment (as well as angular alignments) in a collision can be significant. A point mass reconstruction of a collision simply is incapable of making any such distinctions and can produce results that are considerably different from an actual collision. A reconstructionist must be very careful when applying various methods based on the assumptions that underlie the method. A reconstructionist must also be very careful when choosing software and making claims about a reconstruction. If the software being used to analyze and reconstruct a collision does not take rotation into account, it should be used only under strict adherence to all of the assumptions that govern point mass collisions.

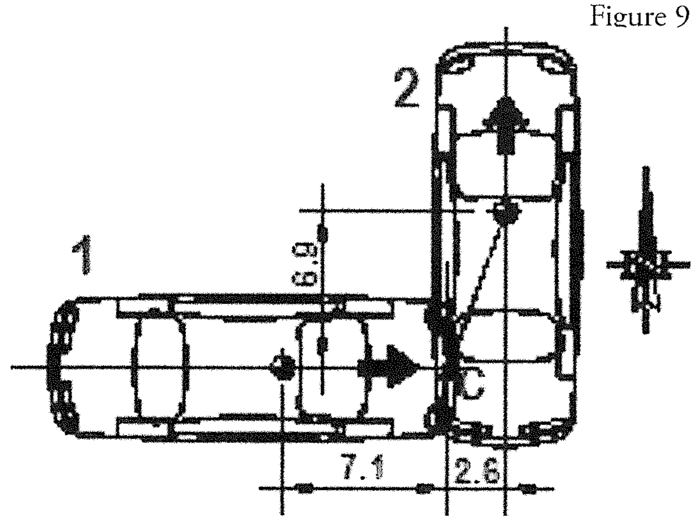
When an impact center, Point C, is located and the angle, Γ, of the crush surface is determined, the planar impact mechanics solution automatically determines the PDOF of each vehicle. The importance of these angles has increased in the advent of Event Data Recorders (EDR). Typically the output from an EDR includes the ΔV_{EDR} value of a vehicle for a collision. Currently, this value is obtained by integration of an accelerometer signal where the accelerometer axis is coincident (aligned) with the heading axis of the vehicle. This means that the ΔV_{EDR} value is not the total vehicle change in speed, ΔV_{VEH} but rather its component along the heading axis. This means that

$$\Delta V_{EDR} = \Delta V_{VEH} \cos(PDOF) \qquad (33)$$

This relationship can be useful in reconstructing accidents when an EDR record is available for one or both vehicles.

COLM should not be used to analyze a collision whenever rotational effects are significant. How can the significance of rotational effects be assessed prior to analysis? Since planar impact mechanics is relatively easy to apply, use both and compare. On the other hand, since planar impact mechanics is easy to apply, it should always be used.

Figure 9



Analysis of a Planar Vehicle Collision

COLmagPIM1.xls
version 2.0
4/13/2007

			Vehicle 1			Vehicle 2
5280/3600	1.467		77.71	mass, m	lb-s^2/ft	77.71
g	32.2	ft/s^2	1575.00	inertia, I	ft-lb-s^2	1575.00
			7.10	distance, d	ft	2.60
e	0.000		0.00	angle, ϕ	deg	90.00
μ (% μ_0)	100.0		180.00	angle, θ	deg	90.00
μ	-0.490					
μ_0	-0.490			INITIAL Velocity		
Γ	0.000	deg	Vehicle 1			Vehicle 2
			61.90	v_x	ft/s	0.00
mbar	38.855		0.00	v_y	ft/s	73.14
k_1^2	20.268	ft^2	0.00	ω	deg/sec	0.00
k_2^2	20.268	ft^2	61.90	v	ft/s	73.14
q	1.000		61.90	v_n	ft/s	0.00
v_{rn}	-61.900	ft/s	0.00	v_t	ft/s	73.14
v_{rt}	73.140	ft/s	61.90	v_{cn}	ft/s	0.00
r	-1.182		0.00	v_{ct}	ft/s	73.14
γ_1	180.000	deg				
γ_2	90.000	deg		FINAL Velocity		
d_a	0.000	ft	Vehicle 1			Vehicle 2
d_b	-2.600	ft	30.95	V_x	ft/s	30.95
d_c	0.000	ft	15.17	V_y	ft/s	57.97
d_d	-7.100	ft	304.52	Ω	deg/sec	111.51
d_e	-3.480	ft	34.47	V	ft/s	65.71
d_f	-1.275	ft	30.95	V_n	ft/s	30.95
A	1.000		15.17	V_t	ft/s	57.97
B	0.000		30.95	V_{cn}	ft/s	30.95
C	1.410		52.91	V_{ct}	ft/s	52.91
D	2.410					

	Initial speeds	
Vehicle 1		Vehicle 2
42.2	mph	49.9
	Final speeds	
Vehicle 1		Vehicle 2
23.5	mph	44.8
	ΔV	
Vehicle 1		Vehicle 2
23.5	mph	23.5
34.5	ft/s	34.5

System Kinetic Energy, ft-lb

Initial	356,730.5	
Final	239,175.5	
Loss	117,555.0	33.0%

Range of Normal (Crush) Energy Loss:			
74,438.6	20.9%	74,438.6	20.9%
Range of Tangential Energy Loss:			
43,116.4	12.1%	43,116.4	12.1%
Total System Energy Loss			
117,555.0	33.0%	117,555.0	33.0%

Impulses, lb-s

P_x	P_y	P
-2405.1	1179.0	2678.6
P_n	P_t	P
-2405.1	1179.0	2678.6

PDOF, deg

Vehicle 1	Vehicle 2
26.1	-63.9

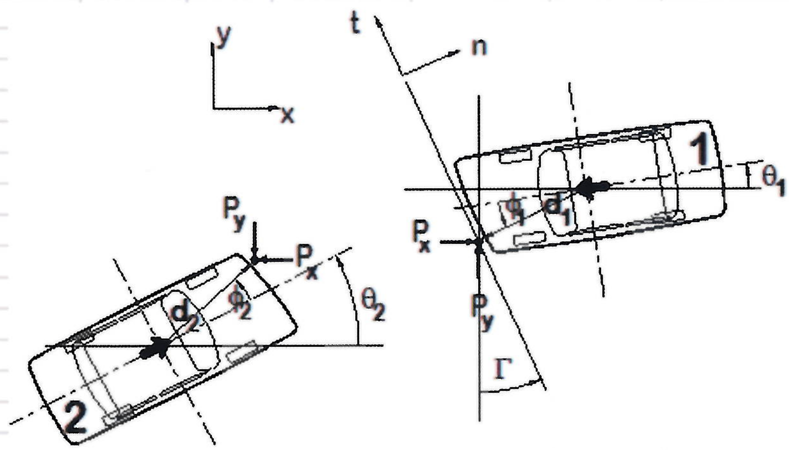


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Figure 8

Analysis of a Planar Vehicle Collision

COLmagPIM2.xls
version 2.0
4/13/2007

5280/3600	1.467	
g	32.2	ft/s^2
e	0.000	
μ (% μ_o)	100.0	
μ	-1.041	
μ_o	-1.041	
Γ	0.000	deg
mbar	38.855	
k_1^2	20.268	ft^2
k_2^2	20.268	ft^2
q	0.411	
v_{rn}	-61.900	ft/s
v_{tn}	73.140	ft/s
r	-1.182	
η_1	180.000	deg
η_2	90.000	deg
d_a	-6.459	ft
d_b	-2.428	ft
d_c	0.000	ft
d_d	-7.100	ft
d_e	-7.390	ft
d_f	-8.986	ft
A	2.029	
B	0.387	
C	1.389	
D	2.389	

Vehicle 1			Vehicle 2
77.71	mass, m	lb-s^2/ft	77.71
1575.00	inertia, I	ft-lb-s^2	1575.00
7.10	distance, d	ft	6.90
0.00	angle, ϕ	deg	159.40
180.00	angle, θ	deg	90.00

INITIAL Velocity

Vehicle 1			Vehicle 2
61.90	v_x	ft/s	0.00
0.00	v_y	ft/s	73.14
0.00	ω	deg/sec	0.00
61.90	v	ft/s	73.14
61.90	v_n	ft/s	0.00
0.00	v_t	ft/s	73.14
61.90	v_{cn}	ft/s	0.00
0.00	v_{ct}	ft/s	73.14

FINAL Velocity

Vehicle 1			Vehicle 2
49.17	V_x	ft/s	12.73
13.25	V_y	ft/s	59.89
265.88	Ω	deg/sec	323.30
50.93	V	ft/s	61.23
49.17	V_n	ft/s	12.73
13.25	V_t	ft/s	59.89
49.17	V_{cn}	ft/s	49.17
46.19	V_{ct}	ft/s	46.19

Vehicle 1		Vehicle 2
	Initial speeds	
42.2	mph	49.9
	Final speeds	
34.7	mph	41.7
	ΔV	
12.5	mph	12.5
18.4	ft/s	18.4

System Kinetic Energy, ft-lb

Initial	356,730.5	
Final	288,473.7	
Loss	68,256.7	19.1%

Range of Normal (Crush) Energy Loss:			
30,611.2	8.6%	27,894.3	7.8%
Range of Tangential Energy Loss:			
37,645.5	10.6%	40,362.4	11.3%
Total System Energy Loss			
68,256.7	19.1%	68,256.7	19.1%

Impulses, lb-s

P_x	P_y	P
-989.1	1029.4	1427.6
P_n	P_t	P
-989.1	1029.4	1427.6

PDOF, deg

Vehicle 1	Vehicle 2
46.1	-43.9

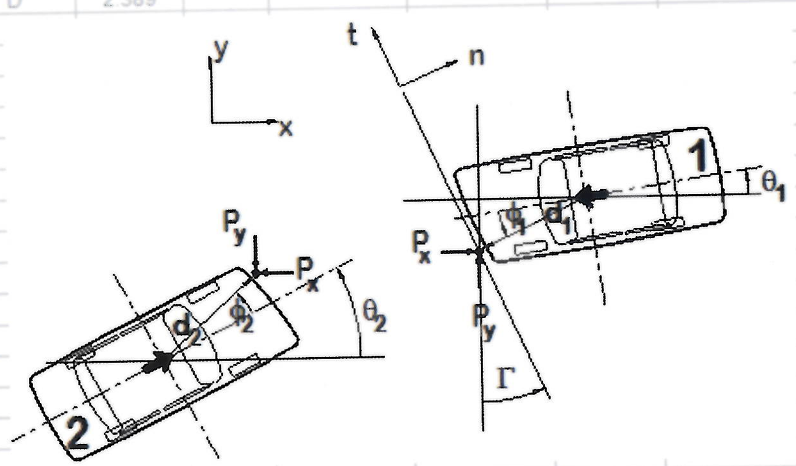

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Figure 10

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Appendix A, Solution of Equations 5 and 6 and Other Point Mass Solution Equations:

$$v_1 = \frac{(m_1V_1\cos\phi_1 + m_2V_2\cos\phi_2)\sin\theta_2 - (m_1V_1\sin\phi_1 + m_2V_2\sin\phi_2)\cos\theta_2}{m_1(\cos\theta_1\sin\theta_2 - \sin\theta_1\cos\theta_2)} \quad \text{(A1)}$$

$$v_2 = \frac{(m_1V_1\cos\phi_1 + m_2V_2\cos\phi_2)\sin\theta_1 - (m_1V_1\sin\phi_1 + m_2V_2\sin\phi_2)\cos\theta_1}{m_2(\cos\theta_2\sin\theta_1 - \sin\theta_2\cos\theta_1)} \quad \text{(A2)}$$

The impulse components in the x-y coordinate system are:

$$P_x = m_1(V_{1x} - v_{1x}) = -m_2(V_{2x} - v_{2x}) \quad \text{(A3)}$$

$$P_y = m_1(V_{1y} - v_{1y}) = -m_2(V_{2y} - v_{2y}) \quad \text{(A4)}$$

The impulse components in the n-t coordinate system are:

$$P_n = P_x\cos\Gamma + P_y\sin\Gamma \quad \text{(A5)}$$

$$P_t = -P_x\sin\Gamma + P_y\cos\Gamma \quad \text{(A6)}$$

The kinetic energy loss of the collision is:

$$KE_{loss} = \frac{1}{2}\bar{m}(v_{2n} - v_{1n})^2(1+e)\left[(1-e) + 2\mu r - (1+e)\mu^2\right] \quad \text{(A7)}$$

where the initial velocity components are:

$$v_{1n} = v_{1x}\cos\Gamma + v_{1y}\sin\Gamma \quad \text{(A8)}$$

$$v_{1t} = -v_{1x}\sin\Gamma + v_{1y}\cos\Gamma \quad \text{(A9)}$$

$$v_{2n} = v_{2x}\cos\Gamma + v_{2y}\sin\Gamma \quad \text{(A10)}$$

$$v_{2t} = -v_{2x}\sin\Gamma + v_{2y}\cos\Gamma \quad \text{(A11)}$$

the final velocity components are:

$$V_{1n} = V_{1x}\cos\Gamma + V_{1y}\sin\Gamma \quad \text{(A12)}$$

$$V_{1t} = -V_{1x}\sin\Gamma + V_{1y}\cos\Gamma \quad \text{(A13)}$$

$$V_{2n} = V_{2x}\cos\Gamma + V_{2y}\sin\Gamma \quad \text{(A14)}$$

$$V_{2t} = -V_{2x}\sin\Gamma + V_{2y}\cos\Gamma \quad \text{(A15)}$$

the impact coefficients are:

$$e = -\frac{V_{2n} - V_{1n}}{v_{2n} - v_{1n}} \quad \text{(A16)}$$

$$\mu = P_t / P_n \quad \text{(A17)}$$

$$\bar{m} = \frac{m_1m_2}{m_1 + m_2} \quad \text{(A18)}$$

$$r = \frac{v_{2t} - v_{1t}}{v_{2n} - v_{1n}} \quad \text{(A19)}$$

APPENDIX B: Notation and Solution Equations of Planar Impact Mechanics

Notation, Subscripts:

cg center of gravity
n, t normal & tangential axes (Fig 2)
x, y ground based axes (Fig 2)
r relative
C impact center
1, 2 vehicle number
P point *P*

Notation, Variables:

C crush
d_0, d_1 crush stiffness coefficients
d_a, d_b distances, Appendix A
d_c, d_d distances, Appendix A
d_e, d_e distances, Appendix A
e coefficient of restitution
E energy
E^* defined in Eq 3
E_C energy loss due to crush
I yaw moment of inertia
K_i constants in Eq 4
m mass
P impulse
r velocity ratio
T kinetic energy
v initial velocity
V final velocity
W work
ΔV velocity change
Γ crush surface angle
ω initial angular velocity
Ω final angular velocity
μ impulse ratio

Summary of assumptions for planar impact mechanics:

1. A single dynamic contact, taking place over a short duration.
2. Forces other than the contact force and impulses of forces other than the contact force are negligible.
3. Rotational motion of the masses can be significant.
4. Initial velocities are known and final velocities are unknown.
5. Deformation is localized and small compared to the size of the bodies.
6. During the contact duration, position and orientation changes are negligibly small, velocity changes are instantaneous and accelerations are large.
7. The effects of the normal (crush) and tangential (sliding, shearing, entanglement, crush, etc.) contact processes are known (through coefficients).
8. A point (impact center), C, common to both vehicles and on the line of action of the contact impulse is known
9. A common crush plane defined by the angle Γ, is known.

Solution Equations of planar impact mechanics:

$$V_{1n} - v_{1n} = \overline{m}(1+e)v_{rn}q / m_1 \quad \text{(B1)}$$

$$V_{1t} - v_{1t} = \mu\overline{m}(1+e)v_{rn}q / m_1 \quad \text{(B2)}$$

$$V_{2n} - v_{2n} = -\overline{m}(1+e)v_{rn}q / m_2 \quad \text{(B3)}$$

$$V_{2t} - v_{2t} = -\mu\overline{m}(1+e)v_{rn}q / m_2 \quad \text{(B4)}$$

$$\Omega_1 - \omega_1 = \overline{m}(1+e)v_{rn}(d_c - \mu d_d)q / (m_1 k_1^2) \quad \text{(B5)}$$

$$\Omega_2 - \omega_2 = \bar{m}(1+e)v_{rn}(d_a - \mu d_b)q/(m_2 k_2^2)$$

$$e = -V_{Crn}/v_{Crn}$$

$$\mu = P_t / P_n$$

$$I_1 = m_1 k_1^2$$

$$I_2 = m_2 k_2^2$$

$$v_{rn} = (v_{2n} - d_a\omega_2) - (v_{1n} - d_c\omega_1)$$

$$V_{Crn} = V_{1n} + d_c\Omega_1 - V_{2n} + d_a\Omega_2$$

$$v_{Crn} = v_{1n} + d_c\omega_1 - v_{2n} + d_a\omega_2$$

$$\frac{1}{q} = 1 + \frac{\bar{m}d_a^2}{m_2k_2^2} + \frac{\bar{m}d_c^2}{m_1k_1^2} - \mu\left(\frac{\bar{m}d_c d_d}{m_1k_1^2} + \frac{\bar{m}d_a d_b}{m_2k_2^2}\right)$$

$$d_a = d_2\sin(\theta_2 + \varphi_2 - \Gamma)$$

$$d_b = d_1\sin(\theta_1 + \varphi_1 - \Gamma)$$

$$d_c = d_1\cos(\theta_1 + \varphi_1 - \Gamma)$$

$$d_d = d_1\cos(\theta_1 + \varphi_1 - \Gamma)$$

$$d_e = d_c - \mu d_d$$

$$d_f = d_a - \mu d_b$$

$$r = \frac{(v_{2t} - d_b\omega_2) - (v_{1t} + d_d\omega_1)}{(v_{2n} - d_a\omega_2) - (v_{1n} + d_c\omega_1)}$$

$$P_x = m_1(V_{1x} - v_{1x})$$

$$P_y = m_1(V_{1y} - v_{1y})$$

$$P_n = P_x\cos\Gamma + P_y\sin\Gamma$$

$$P_t = -P_x\sin\Gamma + P_y\cos\Gamma$$

$$\bar{m} = m_1 m_2/(m_1 + m_2)$$

$$\mu_0 = \frac{rA + (1+e)B}{(1+e)(1+C) + rB}$$

$$A = 1 + \bar{m}(d_c^2/I_1 + d_a^2/I_2)$$

$$B = \bar{m}(d_c d_d/I_1 + d_a d_b/I_2)$$

$$C = \bar{m}(d_d^2/I_1 + d_b^2/I_2)$$

Analysis of Collisions, Conservation of Linear Momentum: Can we do better?

Raymond Brach, University of Notre Dame & Brach Engineering, LLC
Matthew Brach, Brach Engineering, LLC

An article was published in the last issue of *Collision*, "Analysis of Collisions, Point Mass Mechanics and Planar Impact Mechanics", Vol 2, Issue 1. The article summarized some of the main concepts and equations of two dimensional point mass mechanics. Typically referred to as Conservation of Linear Momentum (COLM), the equations commonly are used to reconstruct collisions. The use of COLM doesn't require the value of the coefficient of restitution or the PDOF as input. Rather, these values can be computed as part of the results and, particularly the coefficient of restitution, should be used to assess the realism of solutions. An example showed that if the value and sign of the coefficient of restitution is ignored by the reconstructionist, the result could be a faulty reconstruction. Another important point made is that the use of COLM ignores vehicle rotations. It was shown that ignoring rotations can lead to significant errors in the collision kinetic energy loss, the value and direction of each vehicle's ΔV and each vehicle's principal direction of force (*PDOF*).

When is it best to use COLM?

Never.

After covering point mass mechanics, the article shifted its perspective and presented a complete coverage of Planar Impact Mechanics. In contrast to COLM, planar impact mechanics is a more general method of analyzing and reconstructing collisions in two dimensions. Moreover, the method takes angular momentum of the vehicles into account. Despite the additional complexity of taking the vehicles' dimensions and rotational velocity changes into account, the initial and final velocity components of both colliding vehicles can be computed without a need for numerical solutions. Unfortunately, the solution equations in the article contained typographical errors. They should have appeared as:

$$V_{1n} = v_{1n} + \bar{m}(1+e)v_{rn}q / m_1 \qquad (26)$$

$$V_{1t} = v_{1t} + \mu\bar{m}(1+e)v_{rn}q / m_1 \qquad (27)$$

$$V_{2n} = v_{2n} - \bar{m}(1+e)v_{rn}q / m_2 \qquad (28)$$

$$V_{2t} = v_{2t} - \mu\bar{m}(1+e)v_{rn}q / m_2 \qquad (29)$$

$$\Omega_1 = \omega_1 + \bar{m}(1+e)v_{rn}(d_c - \mu d_d)q / I_1 \qquad (30)$$

$$\Omega_2 = \omega_2 + \bar{m}(1+e)v_{rn}(d_a - \mu d_b)q / I_2 \qquad (31)$$

The velocity components in capital letters (V_{1n}, V_{1t}, V_{2n}, V_{2t}, Ω_1 and Ω_2) are final velocities; small, or lower case letters, (v_{1n}, v_{1t}, v_{2n}, v_{2t}, ω_1 and ω_2) are initial velocities, n and t refer to the normal and tangential coordinates shown in Figure 6. A full set of equations for the planar impact mechanics problem including definitions of all of the related variables such as v_{rn}, q, μ and $\bar{m}$ in Equations 26 through 31 is presented in the original article.

Examples were presented in the article of how differences in the collision configurations of a crash (ignored when using COLM) can affect final angular velocities. The examples went on to show how those differences significantly affect the collision kinetic energy loss, the value and direction of each vehicle's ΔV and each vehicle's *PDOF*. These can be very important in any reconstruction, but can play a critical role when matching a reconstruction to the speed change from an Event Data Recorder (EDR), ΔV_{EDR}. For a vehicle with an axial sensor, the resultant velocity change of the vehicle, ΔV_{VEH}, (calculated using planar impact mechanics) is related to the value recorded by the EDR by

$$\Delta V_{EDR} = \Delta V_{VEH} \cos(PDOF) \qquad (33)$$

Any errors in computing the ΔV and *PDOF* of a vehicle by ignoring rotations, could make a reconstruction using EDR data invalid.

Main conclusions of the article included:

- calculations using planar impact mechanics are more rigorous and accurate that those of COLM,
- planar impact mechanics should always be used to reconstruct a collision rather than COLM,
- reconstructions using planar impact mechanics are easily computed using ordinary spreadsheet methods, MathCAD, MATLAB, etc. using Eq 26 - 31.

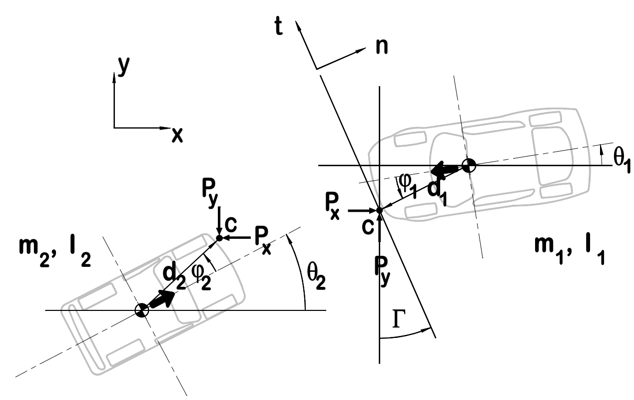

Figure 6. Free body diagrams of two colliding vehicle with coordinate systems and variables.