"Momentum and Energy Analysis of Automobile Collisions", J. Morton, Ed, *Structural Impact and Crashworthiness*, Vol. 2, Imperial College, London, England, July 1984.

MOMENTUM AND ENERGY ANALYSIS OF AUTOMOBILE COLLISIONS

R. M. BRACH

Department of Aerospace and Mechanical Engineering
University of Notre Dame, Notre Dame, Indiana 46556, USA

SYNOPSIS

Principles of Impulse and Momentum frequently are used to study vehicle collisions. In some applications, angular rotations are neglected completely; in others, rotational velocity changes are treated approximately. In this paper, a moment over the crush surface is related to the angular velocity changes and its significance is evaluated with data from experimental collisions. Another feature of Impulse and Momentum models is the treatment of friction. It is shown that a maximum exists for the friction coefficient along with a corresponding maximum kinetic energy loss. This is discussed in general as well as how it leads to a simple equation for velocity change prediction.

1 INTRODUCTION

Many technical papers have appeared which discuss the use of the theory of impulse and momentum for analyzing vehicle collisions. One of the earliest, by Emori[1] in 1968, laid out some of the fundamentals on which later studies could build. Grime and Jones[2], published the most thorough work of its time. Not only were equations developed for vehicle velocity changes, but occupant motion was also treated. Experimental results were provided for vehicle properties, collision energy losses for certain types of collisions and typically encountered parameters such as coefficients of friction and restitution. In 1977, Brach[3] presented a complete set of equations for the planar impact of two vehicles. Basically theoretical, this work yielded six linear algebraic equations which relate the six initial velocity components (three for each vehicle), six final velocity components, vehicle inertial and geometrical properties and the collision geometry. A unique feature of this paper was the recognition that the impulse of a moment must be included over the common area of contact. This led to the definition of a moment coefficient of restitution, e_m. No experimental evidence and only a few values were furnished for this coefficient at that time.

During the mid and late 1970's, a group of experimental collisions was conducted for the U.S. National Highway Traffic Safety Administration[4]. These were a rather comprehensive set of two-vehicle collisions with various configurations, initial speeds and vehicles. These collisions were not only well conducted and well documented but are the first set of collisions which recorded angular velocity changes. In 1983, a paper was published[5] which used the method of least squares to fit the data from the NHTSA collisions to the planar collision model of Brach. This fitting procedure furnished values of the classical restitution coefficient, e, an equivalent friction coefficient, μ , and the moment coefficient e_m. Unfortunately, a sign error crept into the moment equation[3,5] which caused small errors in the coefficient values. The proper results are included in this paper.

Some seemingly strange solutions of the six-equation planar model occur when the equivalent coefficient of friction is varied. Careful examination of this behavior led to some interesting theoretical results of the mechanics of rigid body impacts[6]. Specifically, it was found that a maximum friction coefficient and a corresponding maximum energy loss exist for any given collision. In the work to follow, the concepts of moment impulse, moment coefficient, maximum friction coefficient and maximum energy loss are reviewed. Based upon these concepts, new results are obtained from fitting of the NHTSA collisions to the six-equation planar model.

It was recognized[5] that the velocity change magnitude, ΔV, of each vehicle is predictable by means of an empirical formula. In this paper a theoretical foundation is presented for this formula.

2 PLANAR RIGID BODY IMPACT MECHANICS

Newton's second law states simply that the resultant force on a particle is equal to the product of mass and acceleration. Extensions to rigid body motion and angular rotations are well known. When integrated over an arbitrary time interval the laws relate impulse and change in momentum. The integrated form of Newton's law is convenient for modelling impacts, provided some assumptions are satisfied.

a. The resultant intervehicular impulse is much larger than the impulses of other forces. Thus, during contact, forces such as friction with the ground, drive train drag and aerodynamic drag are neglected.

b. The resultant impulse vector of the intervehicular force (a surface force) acts at a single unique position called the center of impact. The location of this point is assumed to be known.

c. Changes in the position of the mass center and changes in angular orientation are small over the time interval of contact.

d. A hypothetical, fixed contact surface is presumed with the following properties. Motion normal to this surface is due to deformation (crush). Motion parallel to this surface has the nature of relative motion analgous to frictional sliding.

e. The time duration of contact is small.

By itself, Newton's law expressed in impulse/momentum form does not possess any restriction on time duration. Vehicular collisions typically have contact times of 0.1 s to 0.2 s. Time intervals of this magnitude, coupled with the assumption of large forces cause large accelerations, finite velocity changes and small displacements. All of these taken together usually cause the above assumptions to be satisfied for the study of vehicle collisions.

Fig 1 shows free body diagrams of two vehicles. The symbol P represents the resultant vector impulse with components P_x and P_y expressed in the inertial x-y frame. Because the intervehicular impact force acts over a surface, a moment or couple exists. The impulse of the moment is represented by M and is discussed more thoroughly in the next section.

A classical approach to the planar, rigid body impact problem reveals three unknown final velocity components for each rigid body for a total of six. Thus V_{1x}, V_{1y}, Ω_1, V_{2x}, V_{2y}, and Ω_2 are the six unknowns* while the corresponding initial velocity components, v_{1x}, v_{1y}, ω_1, v_{2x}, v_{2y} and ω_2 are presumed known. A proper solution can be obtained with six equations. Since a complete derivation of the six equations of a planar impact has been presented elsehwere,[3,5] along with an analytical solution[6], they are simply listed here in an Appendix.

3 STAGED COLLISIONS AND DATA ANALYSIS

A full set of data is available from 11 collisions[4] which can be grouped into 4 collision catagories. These are illustrated in Fig 2 along with the original collision numbers. Other information such as vehicle physical data, initial speeds, etc., are contained in earlier works[4,5].

---

* Final velocity components are capitalized throughout this paper; initial velocity components are in lower case.

The procedure used to fit the experimental data to the equations[7] in the Appendix is based upon the principle of least squares. It furnishes directly the three coefficients, e, e_m, and μ and corresponding final velocity components. These velocities differ from the experimental values but satisfy conservation of momentum and the other impact equations. Table 1 contains some of the basic results; various detailed aspects of the information contained in this Table are discussed throughout this paper.

4 MOMENT IMPULSE AND MOMENT COEFFICIENT

One novel feature of the impact equations in the Appendix is the provision for treating an impact moment impulse, M. Some authors claim that it is not necessary to introduce this moment impulse into the problem as its presence merely shifts the line of action of the impulse P. Actually, the absence of M can shift the line of action of P away from the true center of impact. A moment impulse can occur during a collision due to momentary or permanent interlocking of parts. A moment impulse can also be created in a collision analysis by a choice of impact center remote from its true location. The moment impulse cannot be omitted indiscriminately. A proper question relates not to the existance of M but rather its significance and its relationship to the moment coefficient.

The definition[3] of the moment coefficient, e_m, allows it to have the values of +1 or $-1 \leq e_m \leq 0$. An alternative, $e' = -e_m$ can be defined[6]. In either case, e_m or e' has properties for rotational velocities which the classical coefficient of restitution, e, has for translational velocities. Specifically, $e_m = +1$ requires that $M = 0$, $e_m = 0$ represents a totally inelastic angular impact ($\Omega_1 = \Omega_2$) and $e_m = -1$ represents a totally elastic angular impact. Both M and e_m depend directly upon the analyst's choice of the location of the impact center, i.e., the location of the point of application of P_x and P_y on each vehicle. These points are typically known with little accuracy. Reference to Fig 1 shows that the distances d_1 and d_2 and angles ϕ_1 and ϕ_2 determine the impact center. For the results in Table 1 d_1, d_2, ϕ_1, and ϕ_2 were chosen with the aid of photographs and scaled deformation diagrams[4]. In order to assess the effects of including a moment in the the impact equations, two sets of least square analyses were conducted. Table 1-A shows results for e_m constrained between -1 and 0 during the fitting procedure. Table 1-B shows the results for $e_m = 1$ (M=0). Generally speaking, the smaller the sum of squares, the better the data fits the equations. For collisions 1, 8, 9 and 10, the condition of $M = 0$ yields a better fit. For all other equations, $M \neq 0$ provides a lower, minimum

TABLE 1 RESULTS FROM ELEVEN EXPERIMENTAL COLLISIONS

A. NON-ZERO MOMENT IMPULSE, M

COLLISION	1	6	7	8	9	10	11	12	3	4	5
MINIMUM SUM OF SQUARES	95.5	101.7	203.3	77.6	275.0	750.7	10.2	31.7	19.2	85.7	106.5
ENERGY LOSS, %	58.2	48.2	48.8	39.5	42.4	43.5	92.2	93.3	34.1	36.3	32.3
RESTITUTION COEFFICIENT	.000	.000	.000	.034	.091	.133	.000	.102	.223	.045	.056
MOMENT COEFFICIENT	.000	-.997	-.815	.000	.000	.000	.000	-.012	-.747	-.517	-.001
FRICTION COEFFICIENT	.922	.872	.778	.500	.809	.837	.025	.020	-.052	-.042	-.071

B. ZERO MOMENT IMPULSE, M

COLLISION	1	6	7	8	9	10	11	12	3	4	5
MINIMUM SUM OF SQUARES	55.4	198.2	221.5	61.3	29.8	100.6	21.8	75.0	29.2	96.2	115.6
ENERGY LOSS, %	52.0	48.2	48.8	36.0	28.9	31.0	90.9	91.9	34.2	36.3	32.0
RESTITUTION COEFFICIENT	.000	.000	.000	.079	.400	.419	.000	.100	.217	.045	.053
MOMENT COEFFICIENT	1	1	1	1	1	1	1	1	1	1	1
FRICTION COEFFICIENT	.966	.824	.772	.413	.486	.590	.038	.031	-.065	-.050	-.090

TABLE 2 VARIATIONS OF ANALYSIS OF COLLISION 6

	Sum of Squares	Energy Loss, %	d_1,m	d_2,m	ϕ_1, deg	ϕ_2, deg	P_x, N-s	P_y, N-s	M, N-m-s	ΔV_1, m/s	ΔV_2, m/s	ΔV_1^*, m/s	ΔV_2^*, m/s	$\Omega_1^* - \Omega_1$, rad/s	$\Omega_2^* - \Omega_2$, rad/s	e	e_m	μ
1	101.7	48.2	2.56	0.61	-17.9	-90.0	447	88	121	4.6	7.6	4.1	6.6	1.35	-0.48	0	-0.997	0.872
2	198.3	48.2	2.56	0.61	-17.9	-90.0	451	75	0	4.6	7.6	4.1	6.6	1.74	-1.60	0	1	0.824
3	145.7	49.4	2.44	0.76	-15.0	-90.0	456	83	4	4.7	7.7	4.1	6.6	1.50	-1.11	0	-0.459	0.849
4	97.2	46.9	2.56	0.69	-25.0	-60.0	468	86	324	4.8	7.9	4.1	6.6	1.09	-0.21	0	-0.709	0.851

* EXPERIMENTAL VALUES

sum of squares. To provide more insight, Collision 6 was analyzed in more detail. A few, unsystematic changes were made in d_1, d_2, ϕ_1 and ϕ_2 and the least square fitting procedure repeated. Results are shown in Table 2. Rows 1 and 2 in Table 2 correspond to the conditions in Table 1 for Collision 6. Rows 3 and 4 are for different impact points.

Another aspect of angular velocities concerns rotational kinetic energy. Note the initial angular velocity of all vehicles in all NHTSA collisions was zero. Rotational energy was as high as 75% of final energy in one case and values of 25% to 30% are typical. So from an energy point of view, rotational velocities are not always negligible.

Based upon comparisons among Tables 1A, 1B, 2 and previous work not reported here and from the point of view of least square fitting of experimental data:

1. Small changes in impact center location do not significantly affect computation of ΔV, the translational velocity change, for each vehicle.
2. Small changes in impact center location do not significantly change the total energy loss of the collision.
3. Small changes in impact center location significantly affect the resulting value of e_m and also affect the other coefficients but to a lesser extent.
4. Omission of the moment impulse can significantly affect the angular velocity changes.

5 MAXIMUM FRICTION COEFFICIENT AND ENERGY LOSS

An analytical solution to the six equations in the Appendix is relatively easy to obtain[6]. The solved equations are long and not easily manipulated however. It is convenient to discuss the concept of energy loss using the point-mass solution and to express the velocities in normal and tangential components. See Fig 1 for the n,t coordinate axes. Conservation of momentum in each coordinate direction gives*

$$m_1V_{1n} + m_2V_{2n} = m_1v_{1n} + m_2v_{2n} \quad \ldots \quad (1)$$

$$m_1V_{1t} + m_2V_{2t} = m_1v_{1t} + m_2v_{2t} \quad \ldots \quad (2)$$

---

* Recall, capital V's represent final velocity components and small, or lower case, v's represent initial velocity components.

The classical coefficient of restitution relates the normal relative velocities by

$$V_{2n} - V_{1n} = -e\,(v_{2n} - v_{1n}) \qquad \ldots \qquad (3)$$

A fourth equation is obtained by expressing the tangential impulse as some proportion of the normal impulse; thus:

$$P_t = \mu P_n \qquad \ldots \qquad (4)$$

The quantity μ is referred to as an equivalent coefficient of friction since actual collisions seldom satisfy the conditions for Coloumb friction. Since impulse equals change in momentum, Eq 4 can be rewritten as:

$$m_1V_{1t} - \mu m_2V_{2n} = m_1v_{1t} - \mu m_2v_{2n} \qquad \ldots \qquad (5)$$

A convenient solution form of these equations is:

$$V_{1n} = v_{1n} + \bar{m}\,(1+e)\,(v_{2n} - v_{1n})/m_1 \qquad \ldots \qquad (6)$$

$$V_{2n} = v_{2n} - \bar{m}\,(1+e)\,(v_{2n} - v_{1n})/m_2 \qquad \ldots \qquad (7)$$

$$V_{1t} = v_{1t} + \mu\bar{m}\,(1+e)\,(v_{2n} - v_{1n})/m_1 \qquad \ldots \qquad (8)$$

$$V_{2t} = v_{2t} - \mu\bar{m}\,(1+e)\,(v_{2n} - v_{1n})/m_2 \qquad \ldots \qquad (9)$$

where $\bar{m} = m_1m_2/(m_1+m_2)$. If $r = (v_{2t} - v_{1n})/(v_{2n} - v_{1n})$ the total kinetic energy loss due to the impact is

$$T_L = \bar{m}(v_{2n} - v_{1n})^2\,(1+e)[(1-e) + 2\mu r - (1+e)\mu^2]/2 \qquad \ldots \qquad (10)$$

This expression has been obtained by many others for the special case of $\mu = 0$. Note that T_L is quadratic in the variable μ; a maximum exists for $\mu = \mu_m$, where

$$\mu_m = r/(1+e) \qquad \ldots \qquad (11)$$

The corresponding maximum energy loss is

$$(T_L)_m = \frac{1}{2}\,\bar{m}\,(v_{2n} - v_{1n})^2\,(1 - e^2 + r^2) \qquad \ldots \qquad (12)$$

The condition $\mu = \mu_m$ corresponds to the case when friction is just large enough to cause zero relative tangential velocity at separation. Any value of $\mu > \mu_m$ (with the use of Eq 4) causes an unrealistically large tangential impulse with a corresponding creation of energy. This accounts for the decrease in T_L for $\mu > \mu_m$ and the maximum behavior. The same behavior exists for the six-equation, rigid body solution.

When the NHTSA data was used to fit the rigid body model, earlier solutions[5] produced a value of μ higher than μ_m by a few percent. Consequently a constraint, $\mu \leq \mu_m$, was added and all eleven collisions were rerun. These are the results in Table 1. Although not illustrated directly in this Table, the least square results show that μ is equal to μ_m for all eleven collisions. This means that for all of the experimental collisions, relative tangential motion ended before separation. One consequence of this is the ability to predict ΔV with a simple formula as shown in the next section.

6 ΔV PREDICTION

For all experimental collisions analyzed it was noted that a certain combination of momentum and energy, denoted by L_i, remained nearly constant[5], where

$$L_i = \frac{m_i \Delta V_i (\Delta T)^{\beta}}{[(m_1 v_1)^2 + (m_2 v_2)^2]^{1/2}} \qquad \ldots \quad (13)$$

Here ΔV_i is the magnitude of the vector velocity change of vehicle with mass m_i; ΔT is the fraction of the total energy loss in the collision and the quantity in the brackets is the magnitude of the total initial momentum. A regression analysis showed the constants L_i and β to be approximately 2/3 and -1/2, respectively, providing the empirical relationship

$$\Delta V_i = L_i [m_1 v_1)^2 + (m_2 v_2)^2]^{1/2} (\Delta T)^{1/2} / m_i \qquad \ldots \quad (14)$$

There appears to be some justification for this relationship based upon two properties of the experimental collisions. Recall an earlier remark that from all of the least square results, the coefficient μ was equal to μ_m, its maximum value. Consider Eq 13 (with β = -1/2) for a point mass collision and for $\Delta T = (\Delta T)_m$; it is

$$L_i^2 = \frac{[r^2 + (1+e)^2](m_1 \bar{m} v_1^2 + m_2 \bar{m} v_2^2)}{[r^2 + (1-e^2)](m_1^2 v_1^2 + m_2^2 v_2^2)} \qquad \ldots \quad (15)$$

where $r = (v_{2t} - v_{1t})/(v_{2n} - v_{1n})$ and $\overline{m} = m_1 m_2/(m_1+m_2)$. Table 1 shows that with only 3 exceptions, the coefficient of restitution is equal to or less than 0.1. With the assumption that $e \sim 0$, L_i can be written as

$$L_i = \left[\frac{\alpha(\alpha + R^2)}{(1+\alpha)(\alpha^2 + R^2)}\right]^{1/2} \qquad \ldots \quad (16)$$

where $\alpha = m_1/m_2$ and $R = v_2/v_1$. Fig 3 shows L_i plotted versus the initial speed ratio, R , with the mass ratio α as a parameter. Note, for all collisions of vehicles with the same mass, $L_i = 0.707$. Deviations from this value for common vehicle mass ratios are relatively small. Under any circumstance[6], Eq 16 can be used with Eq 14 to predict the ΔV for any collision based upon the initial conditions and the energy loss. If the energy loss is unknown, and the collision geometry is similar to the experimental ones, the ΔT's from Table 1 can be used as estimates. Fig 3 shows a comparison plot of ΔV_i for all of the experimental collisions. Actually, 3 sets of ΔV_i's exist, the experimental values, the least square calculated values and predicted values. The predicted values plotted use Eq 14 with $L_i = 2/3$.

An equation simpler than Eq 14 can be obtained from the point mass solution. Eq 6 through 9 with $\mu = \mu_m$ give

$$\Delta V_i = \frac{\bar{m}}{m_i}[(1+e)^2 + r^2]^{1/2}(v_{2n} - v_{1n}) \quad \ldots \quad (17)$$

Though Eq 17 is simpler than Eq 14 and 16, its values for ΔV_i are not as accurate. A reason is that μ_m used in Eq 17 corresponds to the point mass solution and it can vary considerably from the rigid body value of μ_m. As an example, consider Collison 1. The point mass value of $\mu_m = r/(1+e) = 1.73$, whereas $\mu_m = 0.966$ from the least square rigid body solution. Furthermore, the value of ΔT used in Eq 14 is the experimental value.

REFERENCES

1. Emori, R.I., "Analytical Approach to Automobile Collisions," SAE Paper No. 680016, Jan., 1968.
2. Grime, G. and Jones, I., "Car Collisions - The Movement of Cars and Their Occupants in Accidents," Proc. Instn. Mech. Engrs., Vol. 184, Pt 2A, No. 5, 1969-1970.
3. Brach, R.M., "An Impact Moment Coefficient for Vehicle Collision Analysis," Trans. SAE 770014, 1977.
4. Jones, I.S. and Baum, A.S., "Research Input for Computer Simulation of Automobile Collisions, Vol. IV: Staged Collision Reconstructions," DOT HS-805040, Final Report, December, 1978.
5. Brach, R.M., "Impact Analysis of Two-Vehicle Collisions," SAE Paper No 83048 March, 1983.
6. Brach, R.M., "Friction, Restitution and Energy Loss in Planar Collisions," to appear, J. Appl. Mech., Trans. ASME, 1984
7. Brach, R.M., "Nonlinear Estimation of a Vehicle Collision Model," Proc. 13th Modeling and Simulation Conf., University of Pittsburgh, April, 1982.

APPENDIX - EQUATIONS OF IMPULSE/MOMENTUM MODEL

Conservation of momentum along the x axis: $m_2(V_{2x}-v_{2x}) + m_1(V_{1x}-v_{1x}) = 0$

Conservation of momentum along the y axis: $m_2(V_{2y}-v_{2y}) + m_1(V_{1y}-v_{1y}) = 0$

Conservation of angular momentum: $I_2(\Omega_2-\omega_2) + I_1(\Omega_1-\omega_1) + m_2(d_a+d_c)(V_{2x}-v_{2x})$
$+ m_1(d_b+d_d)(v_{1y}-v_{1y}) = 0$

Restitution normal to the crush line at angle Γ: $(V_{1y}-d_d\Omega_1-V_{2y}-d_b\Omega_2)\sin\Gamma$
$+ (V_{1x}+d_c\Omega_1-V_{2x}+d_a\Omega_2)\cos\Gamma = -e[(v_{1y}-d_d\omega_1-v_{2y}-d_b\omega_2)\sin\Gamma$
$+ (v_{1x}+d_c\omega_1-v_{2x}+d_a\omega_2)\cos\Gamma$

Friction along the crush line at angle Γ: $m_1(V_{1y}-v_{1y})(\cos\Gamma - \mu\sin\Gamma)$
$+ m_2(V_{2x}-v_{2x})(\sin\Gamma + \mu\cos\Gamma) = 0$

Moment restitution at impact surface: $(\Omega_2-\Omega_1)(1-e_m) = e_m[(\Omega_1-\omega_1)-m_1d_c(V_{1x}-v_{1x})/I_1$
$+ m_1d_d(V_{1y}-v_{1y})/I_1 - (\Omega_2-\omega_2)-m_2d_a(V_{2x}-v_{2x})/I_2$
$+ m_2d_b\ (V_{2y}-v_{2y})/I_2]$

In the above: $d_a = d_2\sin(\Theta_2+\phi_2)$ $d_b = d_2\cos(\Theta_2+\phi_2)$
$d_c = d_1\sin(\Theta_1\ \phi_1)$ $d_d = d_1\cos(\Theta_1+\phi_1)$

NOTATION

e	coefficient of restitution
e_m	moment coefficient of restitution
d	distance between mass center and impact center
I	vehicle yaw inertia about its mass center
m	mass of vehicle
T	kinetic energy
V,v	velocity
μ	equivalent coefficient of friction along the impact surface
Θ	heading angle of vehicles relative to the x axis
Γ	angle of impact surface relative to the y axis
Ω,ω	angular velocity
ϕ	angle between the length axis of a vehicle and a line between its center of gravity and the center of impact

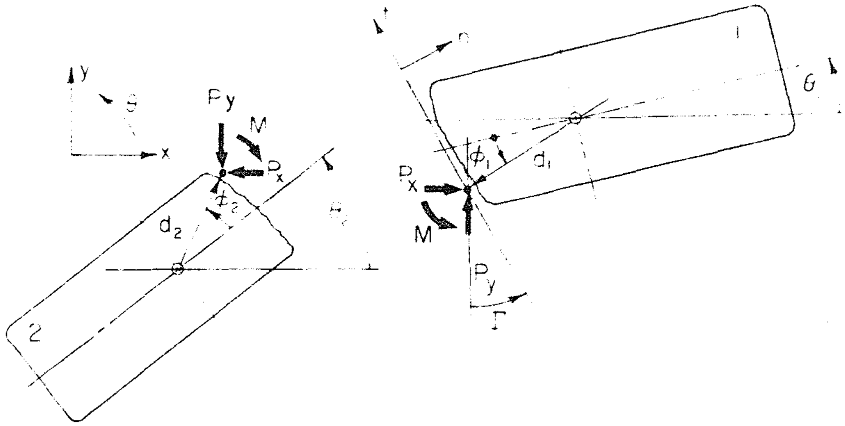

Fig 1   Free Body Diagrams

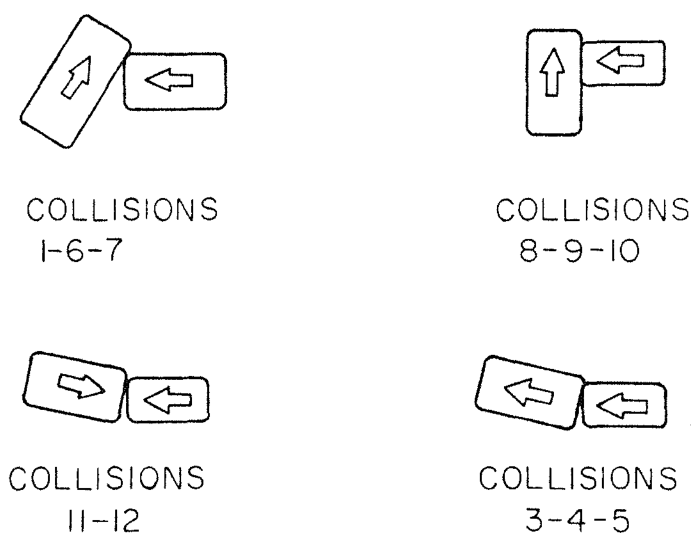

Fig 2   Experimental Collision Configurations

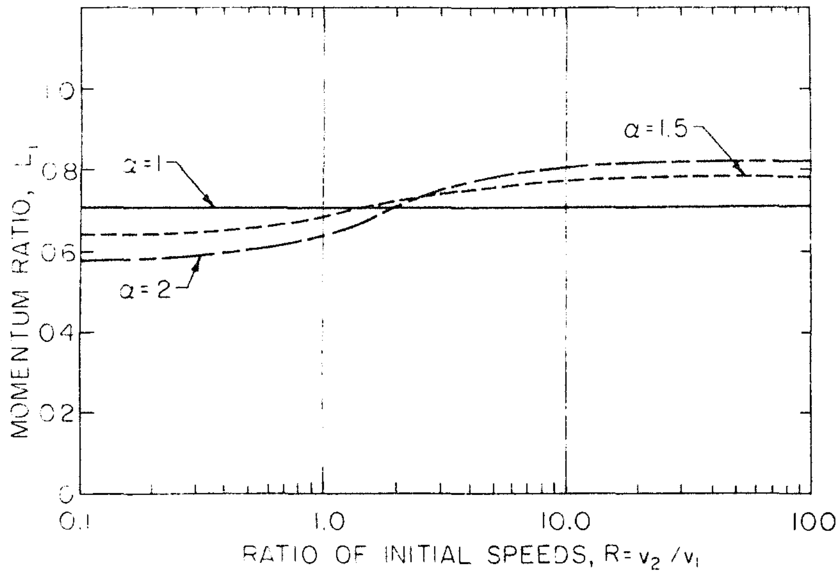

Fig 3 Coefficient L_i for ΔV Prediction (Eq 14)

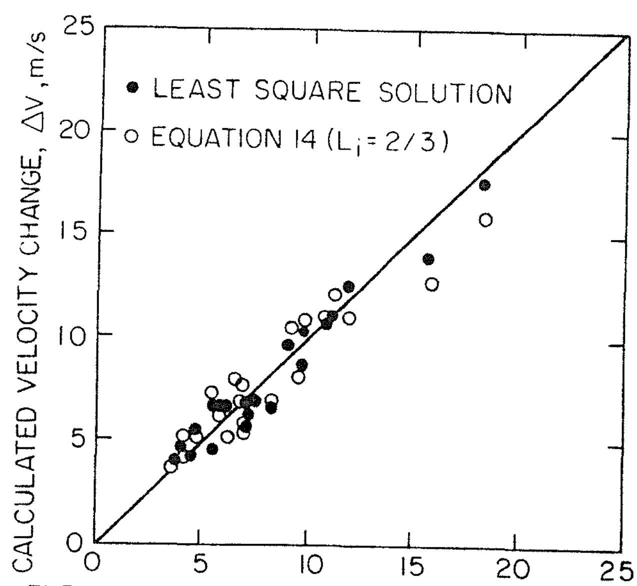

Fig 4 Comparison of Calculated Velocity Changes with Experimental Values